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CITATION:

Katagata, Koh. Julia sets of quartic polynomials and a topology of the symbol space(Complex Dynamics and its Related Topics). 数理解析研究所講究録 2007, 1537: 155-162

ISSUE DATE:

2007-02

URL:

<http://hdl.handle.net/2433/59024>

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# Julia sets of quartic polynomials and a topology of the symbol space

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## Abstract

For a certain quartic polynomial, there exists a homeomorphism between the set of all components of the filled-in Julia set with the Hausdorff metric and some subset of the corresponding symbol space with the ordinary metric known well. But these sets are not compact with respect to each metric. We introduce new topologies with respect to which these sets are compact.

## 1 Introduction and the main results

Let  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  be the Riemann sphere and let  $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  be a rational function of degree  $d \geq 2$ . In the theory of the complex dynamics, there are two important sets called the Fatou set  $F(f)$  and the Julia set  $J(f)$ . The Fatou set  $F(f)$  is the set of normality in the sense of Montel for the family  $\{f^n\}_{n=0}^{\infty}$ , where  $f^n = f \circ \cdots \circ f$  is  $n$  iterates of  $f$ . The Julia set  $J(f)$  is the complement  $\hat{\mathbb{C}} \setminus F(f)$ .  $J(f)$  is either connected or else has uncountably many connected components. In the case that  $f$  is a polynomial, we define the filled-in Julia set  $K(f)$  as

$$K(f) = \{z \in \mathbb{C} : \{f^n(z)\}_{n=0}^{\infty} \text{ is bounded}\}.$$

$J(f)$  is the topological boundary of  $K(f)$ . We call  $A(f) = \hat{\mathbb{C}} \setminus K(f)$  the attracting basin of the point at infinity.

We often consider another model in order to simplify dynamics of  $f$ . The model is the symbol space and the shift map defines a dynamical system on the symbol space. Let  $X^{\omega}$  be the countable product of a set  $X$ .

**Definition 1.1.** The *symbol space* of  $q$ -symbols is the countable product  $\Sigma_q = \{1, 2, \dots, q\}^\omega$ . For  $s = (s_n)$  and  $t = (t_n) \in \Sigma_q$ , a metric  $\rho$  on  $\Sigma_q$  is defined as

$$\rho(s, t) = \sum_{n=0}^{\infty} \frac{\delta(s_n, t_n)}{2^n}, \quad \text{where } \delta(k, l) = \begin{cases} 1 & \text{if } k \neq l, \\ 0 & \text{if } k = l. \end{cases}$$

Then  $(\Sigma_q, \rho)$  is a compact metric space. The *shift map*  $\sigma : \Sigma_q \rightarrow \Sigma_q$  is defined as

$$\sigma((s_0, s_1, s_2, \dots)) = (s_1, s_2, \dots).$$

The shift map  $\sigma$  is continuous with respect to the metric  $\rho$ .

The connectivity of the Julia set of a polynomial of degree two or more is affected by the behavior of finite critical points.

**Theorem 1.2.** *Let  $f$  be a polynomial of degree  $d \geq 2$ . If all finite critical points of  $f$  are in  $A(f)$ , then  $J(f)$  is totally disconnected. Furthermore  $f|_{J(f)}$  is topologically conjugate to the shift map  $\sigma : \Sigma_d \rightarrow \Sigma_d$ . On the other hand,  $J(f)$  is connected if and only if all finite critical points of  $f$  are in  $K(f)$ .*

If some critical orbits of a polynomial converge to the point at infinity but all critical orbits do not converge to it, then the Julia set is disconnected and not generally totally disconnected. It is a problem whether dynamics of a polynomial on the Julia set can be simplified as dynamics of the shift map on some symbol space when the Julia set is disconnected and not totally disconnected. But it is difficult to make points of non-trivial connected components of the Julia set correspond to points of some symbol space. Therefore we consider the set of all components of the Julia set and make it correspond to some symbol space. On account of the following arguments, we consider not the set of all components of the Julia set but the set of all components of the filled-in Julia set. But nothing essentially changes.

**Definition 1.3.** Let  $f$  be a polynomial of degree  $d \geq 2$ . The *Green's function* associated with  $K(f)$  is defined as

$$G(z) = \lim_{n \rightarrow \infty} \frac{1}{d^n} \log^+ |f^n(z)|,$$

where  $\log^+ x = \max \{\log x, 0\}$ .  $G(z)$  is zero for  $z \in K(f)$  and  $G(z)$  is positive for  $z \in \mathbb{C} \setminus K(f)$ . Note that  $G$  satisfies the identity  $G(f(z)) = d \cdot G(z)$ .

**Definition 1.4.** The triple  $(f, U, V)$  is a *polynomial-like map* of degree  $d$  if  $U$  and  $V$  are topological disks with  $\bar{U} \subset V$  and  $f : U \rightarrow V$  is a holomorphic proper map of degree  $d$ . We define the *filled-in Julia set*  $K(f)$  of a polynomial-like map  $(f, U, V)$  as

$$K(f) = \{z \in U : \{f^n(z)\}_{n=0}^{\infty} \subset U\}.$$

**Definition 1.5.** Let  $X$  be a metric space. For a compact subset  $A \subset X$  and  $\delta > 0$ , let  $A[\delta]$  be a  $\delta$ -neighborhood of  $A$ . For compact subsets  $A$  and  $B \subset X$ , we define the *Hausdorff metric*  $d_H$  as

$$d_H(A, B) = \inf\{\delta : A \subset B[\delta] \text{ and } B \subset A[\delta]\}.$$

Let  $f$  be a quartic polynomial and let  $c_1, c_2$  and  $c_3$  be finite critical points of  $f$ . Suppose that  $c_1$  and  $c_2$  are in  $K(f)$  and  $c_3$  is in  $A(f)$ . Let  $U$  be a bounded component of  $\mathbb{C} \setminus G^{-1}(G(f(c_3)))$  and let  $U_A$  and  $U_B$  be bounded components of  $\mathbb{C} \setminus G^{-1}(G(c_3))$ . In other words,  $U = \{z \in \mathbb{C} : G(z) < G(f(c_3))\}$  and  $U_A \cup U_B = \{z \in \mathbb{C} : G(z) < G(c_3)\}$ , where  $G$  is the Green's function associated with  $K(f)$ . Suppose that  $c_1$  is in  $U_A$  and  $c_2$  is in  $U_B$ . Then  $(f|_{U_A}, U_A, U)$  and  $(f|_{U_B}, U_B, U)$  are polynomial-like maps of degree 2. Suppose that filled-in Julia sets  $K_A = K(f|_{U_A})$  and  $K_B = K(f|_{U_B})$  are connected.

Let  $K(f)^*$  be the set of all components of  $K(f)$ . Since  $c_3$  is in  $A(f)$ ,  $K(f)^*$  is uncountable.  $K(f)^*$  becomes a metric space with the Hausdorff metric  $d_H$ . We define a map  $F : (K(f)^*, d_H) \rightarrow (K(f)^*, d_H)$  as  $F(K) = f(K)$  for  $K \in K(f)^*$ . Then  $F$  is continuous.

Let  $\Sigma_6 = \{1, 2, 3, 4, A, B\}^\omega$  be the symbol space. We define a subset  $\Sigma$  of  $\Sigma_6$  as follows:  $s = (s_n) \in \Sigma$  if and only if

- (S1) if  $s_n = A$ , then  $s_{n+1} = A$ ,
- (S2) if  $s_n = B$ , then  $s_{n+1} = B$ ,
- (S3) if  $s_n = A$  and  $s_{n-1} \neq A$ , then  $s_{n-1} = 3$  or  $4$ ,
- (S4) if  $s_n = B$  and  $s_{n-1} \neq B$ , then  $s_{n-1} = 1$  or  $2$ ,
- (S5) if  $s \in \Sigma_4 = \{1, 2, 3, 4\}^\omega$ , then there exist subsequences  $(s_{n(k)})_{k=1}^{\infty}$  and  $(s'_{n(l)})_{l=1}^{\infty}$  such that  $s_{n(k)} = 1$  or  $2$  for all  $k \geq 1$  and  $s'_{n(l)} = 3$  or  $4$  for all  $l \geq 1$ .

The author proved the following theorem in [Ka].

**Theorem 1.6.** *Suppose that a quartic polynomial  $f$ , its finite critical points  $c_1, c_2, c_3$  and domains  $U, U_A, U_B$  are as above. And suppose that filled-in Julia sets  $K_A$  and  $K_B$  are connected. Then there exists a homeomorphism  $\Lambda : (K(f)^*, d_H) \rightarrow (\Sigma, \rho)$  such that  $\Lambda \circ F = \sigma \circ \Lambda$ .*

Theorem 1.6 means that componentwise dynamics of  $f$  on  $K(f)$  (of course, also on  $J(f)$ ) be simplified as dynamics of the shift map on  $\Sigma$ . But  $(\Sigma, \rho)$  is not compact. For example, a sequence

$$\left\{ s^{(n)} = \underbrace{(1, 1, \dots, 1)}_{n \text{ times}}, B, B, \dots \right\}_{n=0}^{\infty}$$

in  $\Sigma$  converges to  $s = (1, 1, 1, \dots)$  but  $s$  is not in  $\Sigma$ . Can we define a topology of  $\Sigma$  with respect to which the sequence  $\{s^{(n)}\}_{n=0}^{\infty}$  converges to a “appropriate” point in  $\Sigma$ , and furthermore  $\Sigma$  is compact? The meaning of “appropriate” is explained in another section. In this paper we answer the question:

**Theorem 1.7.** *Let  $\Sigma$  be as above. Then there exists a topology  $\mathcal{O}$  of  $\Sigma$  such that  $(\Sigma, \mathcal{O})$  is compact, metrizable, perfect and totally disconnected. Moreover the shift map  $\sigma : (\Sigma, \mathcal{O}) \rightarrow (\Sigma, \mathcal{O})$  is continuous.*

By Theorem 1.6, there exists a homeomorphism  $\Lambda : (K(f)^*, d_H) \rightarrow (\Sigma, \rho)$  such that  $\Lambda \circ F = \sigma \circ \Lambda$ . Especially  $\Lambda^{-1} : \Sigma \rightarrow K(f)^*$  is bijective. Let  $\mathcal{G}$  be the quotient topology of  $K(f)^*$  relative to  $\Lambda^{-1}$  and the topology  $\mathcal{O}$  of  $\Sigma$  in Theorem 1.7, that is,

$$\mathcal{G} = \{G \subset K(f)^* : \Lambda(G) \in \mathcal{O}\}.$$

Then  $\Lambda : (K(f)^*, \mathcal{G}) \rightarrow (\Sigma, \mathcal{O})$  is a homeomorphism such that  $\Lambda \circ F = \sigma \circ \Lambda$ .

**Corollary 1.8.**  *$(K(f)^*, \mathcal{G})$  is compact, metrizable, perfect and totally disconnected. Moreover  $F : (K(f)^*, \mathcal{G}) \rightarrow (K(f)^*, \mathcal{G})$  is continuous.*

## 2 Definition of a new topology of $\Sigma$

We define a topology of  $\Sigma$ . If  $s = (A, A, A, \dots) \in \Sigma$ , we define subsets  $N_s^{(k)}$  of  $\Sigma$  as

$$N_s^{(k)} = \{s\} \cup \{t = (t_n) \in \Sigma : t_n = 1 \text{ or } 2 \text{ for } n \leq k\}.$$

Similarly, if  $s = (B, B, B, \dots) \in \Sigma$ ,

$$N_s^{(k)} = \{s\} \cup \{t = (t_n) \in \Sigma : t_n = 3 \text{ or } 4 \text{ for } n \leq k\}.$$

If  $s = (s_0, \dots, s_l, A, A, A, \dots) \in \Sigma$  with  $s_l \neq A$ ,

$$N_s^{(k)} = \{s\} \cup \left\{ t = (t_n) \in \Sigma : t_n = \begin{cases} s_n & \text{if } n \leq l, \\ 1 \text{ or } 2 & \text{if } l+1 \leq n \end{cases} \text{ for } n \leq k \right\}.$$

Similarly, if  $s = (s_0, \dots, s_l, B, B, B, \dots) \in \Sigma$  with  $s_l \neq B$ ,

$$N_s^{(k)} = \{s\} \cup \left\{ t = (t_n) \in \Sigma : t_n = \begin{cases} s_n & \text{if } n \leq l, \\ 3 \text{ or } 4 & \text{if } l+1 \leq n \end{cases} \text{ for } n \leq k \right\}.$$

Finally, if  $s = (s_n) \in \Sigma \cap \Sigma_4$ ,

$$N_s^{(k)} = \{t = (t_n) \in \Sigma : t_n = s_n \text{ for } n \leq k\}.$$

Note that  $N_s^{(k+1)} \subset N_s^{(k)}$  for all  $s \in \Sigma$  and  $k \geq 0$ . Let  $\mathcal{N}(s) = \{N_s^{(k)}\}_{k=0}^\infty$  and  $\mathcal{N} = \{\mathcal{N}(s) : s \in \Sigma\}$ . Then  $\mathcal{N}$  is a neighborhood system of  $\Sigma$  and hence  $(\Sigma, \mathcal{O})$  is a topological space, where

$$\mathcal{O} = \{O \subset \Sigma : \text{if } s \in O, \text{ then there exists } N \in \mathcal{N}(s) \text{ such that } N \subset O\}.$$

The topology  $\mathcal{O}$  satisfies Theorem 1.7.

### 3 Appropriateness of the convergence with respect to $\mathcal{O}$

We can formulate  $\Lambda : K(f)^* \rightarrow \Sigma$  concretely. Refer to [Ka] for the detailed proof. Let  $U, U_A, U_B, K_A$  and  $K_B$  be the same as the section 1. There exist forward invariant rays  $R_{A1}$  and  $R_{B2}$  under  $f$  such that  $R_{A1}$  lands at a point on  $\partial K_A$  and  $R_{B1}$  lands at a point on  $\partial K_B$ . These landing points are repelling or parabolic fixed points of  $f$ . Let  $R_{A2}$  and  $R_{B2}$  be components of  $f^{-1}(R_{A1})$  and  $f^{-1}(R_{B1})$  which satisfy  $R_{A2} \cap U_A \neq \emptyset$  and  $R_{B2} \cap U_B \neq \emptyset$  and differ from  $R_{A1}$  and  $R_{B1}$  respectively. We set  $V_A = U \setminus (K_A \cup R_{A1})$  and  $V_B = U \setminus (K_B \cup R_{B1})$ . Let  $I_1, I_2, I_3$  and  $I_4$  be branches of  $f^{-1}$  such that

$$\begin{aligned} I_1 : V_A &\rightarrow U_1, & I_2 : V_A &\rightarrow U_2, \\ I_3 : V_B &\rightarrow U_3, & I_4 : V_B &\rightarrow U_4, \end{aligned}$$

where  $U_1$  and  $U_2$  are components of  $U_A \setminus K_A \cup R_{A1} \cup R_{A2}$  respectively. Similarly,  $U_3$  and  $U_4$  are components of  $U_B \setminus K_B \cup R_{B1} \cup R_{B2}$  respectively. We define  $\Lambda : K(f)^* \rightarrow \Sigma$  as follows: for  $K \in K(f)^*$ ,

$$[\Lambda(K)]_n = \begin{cases} i & \text{if } f^n(K) \subset U_i, \\ A & \text{if } f^n(K) = K_A, \\ B & \text{if } f^n(K) = K_B, \end{cases}$$

where  $n \geq 0$  and  $i = 1, 2, 3, 4$ . We can also formulate  $\Lambda^{-1} : \Sigma \rightarrow K(f)^*$  as follows: if  $s_n = A$  and  $s_{n-1} \neq A$ ,

$$\Lambda^{-1}(s) = I_{s_0} \circ \cdots \circ I_{s_{n-1}}(K_A).$$

If  $s_n = B$  and  $s_{n-1} \neq B$ ,

$$\Lambda^{-1}(s) = I_{s_0} \circ \cdots \circ I_{s_{n-1}}(K_B).$$

If  $s \in \Sigma_4$ , there exists a subsequence  $(s_{n(l)})_{l=1}^{\infty}$  such that  $s_{n(l)} = 1$  or  $2$  and  $s_{n(l)-1} = 3$  or  $4$ . We set  $K_s^{(l)} = I_{s_0} \circ \cdots \circ I_{s_{n(l)-1}}(\overline{U_A})$ . Then  $K_s^{(l+1)} \subset K_s^{(l)}$  and

$$\Lambda^{-1}(s) = \bigcap_{l=1}^{\infty} K_s^{(l)}.$$

Note that  $\bigcap_{l=1}^{\infty} K_s^{(l)}$  is a one-point set since each  $I_k$  decreases the Poincaré distance on  $V_A$  or  $V_B$ .

We reconsider the sequence

$$\left\{ s^{(n)} = (\underbrace{1, 1, \dots, 1}_{n \text{ times}}, B, B, \dots) \right\}_{n=0}^{\infty}$$

in  $\Sigma$ . It converges to  $s = (1, 1, 1, \dots) \notin \Sigma$  with respect to  $\rho$ . However, it converges to  $s = (A, A, A, \dots) \in \Sigma$  with respect to  $\mathcal{O}$ . We check that the convergence with respect to  $\mathcal{O}$  is "appropriate". By definition of  $\Lambda^{-1}$ ,

$$\Lambda^{-1}(s^{(n)}) = \underbrace{I_1 \circ \cdots \circ I_1}_{n \text{ times}}(K_B).$$

Let  $K^{(n)} = \Lambda^{-1}(s^{(n)})$ . Since  $I_1$  decreases the Poincaré distance on  $V_A$ , the sequence  $\{K^{(n)}\}_{n=0}^{\infty} \subset K(f)^*$  converges to not  $K_A \in K(f)^*$  but a one-point set  $K = \{\zeta\}$  with respect to the Hausdorff metric  $d_H$ . The point  $\zeta$  is actually in  $\partial K_A$ , and therefore  $K \notin K(f)^*$ . We expect that  $\{K^{(n)}\}_{n=0}^{\infty}$  converges to  $K_A$  with respect to  $\mathcal{G}$ . In fact,

$$\lim_{n \rightarrow \infty} K^{(n)} = \lim_{n \rightarrow \infty} \Lambda^{-1}(s^{(n)}) = \Lambda^{-1} \left( \lim_{n \rightarrow \infty} s^{(n)} \right) = \Lambda^{-1}(s) = K_A$$

since  $\Lambda^{-1} : (\Sigma, \mathcal{O}) \rightarrow (K(f)^*, \mathcal{G})$  is continuous. Therefore we express that the convergence of  $\{s^{(n)}\}_{n=0}^{\infty}$  with respect to  $\mathcal{O}$  is "appropriate" in the sense that  $\{K^{(n)}\}_{n=0}^{\infty}$  converges to  $K_A$  with respect to  $\mathcal{G}$ .

## 4 Applications

For a rational function of degree at least two, the backward orbit of a point in the Julia set and the set of all repelling periodic points are dense in the Julia set:

**Theorem 4.1.** *Let  $g$  be a rational function of degree at least two. If  $z \in J(g)$ , then*

$$J(g) = \overline{\bigcup_{k=1}^{\infty} g^{-k}(z)}.$$

**Theorem 4.2.** *Let  $g$  be a rational function of degree at least two. Then*

$$J(g) = \overline{\{\text{repelling periodic point of } g\}}.$$

We obtain analogies of Theorem 4.1 and 4.2.

**Theorem 4.3.** *Let  $(\Sigma, \mathcal{O})$  be as in Theorem 1.7 and let  $s \in \Sigma$ . Then*

$$\Sigma = \overline{\bigcup_{k=1}^{\infty} \sigma^{-k}(s)},$$

where the closure is taken in  $(\Sigma, \mathcal{O})$ .

**Remark 4.4.** The closure of the backward orbit of  $s \in \Sigma$  under  $\sigma$  does not necessarily coincide with  $\Sigma$  in  $(\Sigma, \rho)$ . For example,

$$(A, A, A, \dots) \notin \overline{\bigcup_{k=1}^{\infty} \sigma^{-k}((B, B, B, \dots))},$$

where the closure is taken in  $(\Sigma, \rho)$ .

**Corollary 4.5.** *Let  $(K(f)^*, \mathcal{G})$  be as in Corollary 1.8 and let  $K \in K(f)^*$ . Then*

$$K(f)^* = \overline{\bigcup_{k=1}^{\infty} F^{-k}(K)},$$

where the closure is taken in  $(K(f)^*, \mathcal{G})$ .

**Theorem 4.6.** *Let  $(\Sigma, \mathcal{O})$  be as in Theorem 1.7. Then*

$$\Sigma = \overline{\{\text{periodic point of } \sigma \text{ in } \Sigma\}},$$

where the closure is taken in  $(\Sigma, \mathcal{O})$ .



*Remark 4.7.* The closure of the set of all periodic points of  $\Sigma$  does not coincide with  $\Sigma$  in  $(\Sigma, \rho)$  since  $t = (t_0, t_1, \dots, t_l, A, A, A, \dots)$  with  $t_l \neq A$  is an isolated point in  $(\Sigma, \rho)$ .

**Corollary 4.8.** *Let  $(K(f)^*, \mathcal{G})$  be as in Corollary 1.8. Then*

$$K(f)^* = \overline{\{\text{periodic point of } F \text{ in } K(f)^*\}},$$

*where the closure is taken in  $(K(f)^*, \mathcal{G})$ .*

## References

- [Ka] K. Katagata, On a certain kind of polynomials of degree 4 with disconnected Julia set, submitted
- [Ke] J. L. Kelly, *General Topology*, Springer, 1975.
- [Mi] J. Milnor, *Dynamics in One Complex Variable*, Vieweg, 2nd edition, 2000.
- [Na] J. Nagata, *Modern General Topology*, American Elsevier Publishing Company, 2nd revised edition, 1974.